numbers using the vorticity equation and the results show that a higher heat transfer rate is obtained with an increase in the starting length Reynolds number.

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Heat transfer through a vertical enclosure with convective boundary conditions

V. M. MARBALLI, S. A. KORPELA, Y. LEE and S. NAKAMURA
Department of Mechanical Engineering, The Ohio State University, Columbus, OH 43210, U.S.A.

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1. INTRODUCTION

IN THIS note we discuss the heat transfer through a vertical enclosure with convective boundary conditions. In a previous article we have reviewed some of the history of this problem and considered its relevance to double pane windows [1]. Elsewhere we have discussed the ramifications of replacing the air in the enclosed space by other fluids [2], and how the flow changes when the cavity is an enclosed annulus rather than one of Cartesian geometry [3].

In ref. [1] we took the sidewalls to be isothermal and considered the top and the bottom to be insulated. In this article we report numerical calculations for a more realistic model of a double pane window by taking into account the thermal interaction of the sidewalls with the surroundings. We do this by specifying some reasonable values for the outside heat transfer coefficients.

The aim of this study is to see how the sidewall temperatures vary with height, and how this variation influences the convection in the air gap and heat transfer through it. An important practical outcome of the calculations is that one can estimate the error made if the panes are assumed to be isothermal. Although we have neglected the vertical conduction of heat in the panes, this does not invalidate the analysis, because vertical conduction acts to smooth out temperature variations. As a result the error estimate is a conservative one and provides a bound for what can be expected in practice.

Our calculations are for a cavity with an aspect ratio equal to 20. This is sufficiently high for the flow to undergo a transition to a multicellular convective pattern as the Grashof number is increased to a value of the order of 10 000. It is still, however, too small to be relevant to tall windows. The results in ref. [1] give the optimum gap width for tall windows, thus we are excused for considering them further in this note. We also discuss in Section 3 the qualitative differences which are manifest when a constant wall heat flux is imposed on one wall, while the other is held at a fixed temperature.

2. ANALYSIS

Consider a rectangular region of aspect ratio $A = H/L$ in which H is the height and L is the width. The depth is sufficiently large for the flow to be two-dimensional. The LHS is exposed to an ambient at temperature T_h via a heat transfer coefficient h_l and the RHS interacts with a colder ambient at temperature T_c, with a heat transfer coefficient h_r between them. The temperature differences are assumed to be sufficiently small that the Boussinesq approximation is valid.

The equations and boundary conditions governing convection in this space are given in ref. [2] and are not repeated here, except for the sidewall thermal boundary conditions. These are

$$\frac{\partial T}{\partial x} + Nu_{lL}(1-T) = 0, \quad x = 0 \tag{1}$$

and

$$\frac{\partial T}{\partial x} + Nu_{rL}\,T = 0, \quad x = 1. \tag{2}$$

In equations (1) and (2) the two Nusselt numbers are $Nu_{lL} = h_l L/k$ and $Nu_{rL} = h_r L/k$. These are related to Nusselt numbers based on the height of the cavity via the expressions $Nu_l = h_l H/k = Nu_{lL}A$ and $Nu_r = h_r H/k = Nu_{rL}A$. These Nusselt numbers are input parameters to the computations. We need to define additional Nusselt numbers in order to report the results. The vertically averaged Nusselt number is $Nu = \bar{q}L/k\Delta T$, where $\bar{q}$ is the average heat flux across either the left or the right wall. If we use instead of the overall ΔT, an average temperature difference between the sidewalls, we can define $Nu_w = \bar{q}L/k\Delta\bar{T}_w$, which can also be expressed as $Nu_w = k_{eff}/k$. Here k_{eff} is the effective thermal conductivity of the air gap which can be used in a Fourier type of expression

$$\bar{q} = k_{eff}(\bar{T}_{wl} - \bar{T}_{wr})/L$$

where $\bar{T}_{wl}$ and $\bar{T}_{wr}$ are the vertically averaged wall temperatures.

In this study we are using a somewhat different numerical scheme than we used in refs. [1–3]. It is well known that convective boundary conditions cause numerical methods to become less stable. For this reason, rather than use centered time differences, we opted for a method called a rationalized Runge–Kutta scheme [4]. It has been used by Satofuka [5] for solving the Navier–Stokes equations. We differenced the Jacobian by the Arakawa scheme as before, but now instead of the DuFort–Frankel method for the diffusive terms, we used central differences. Satofuka [5] demonstrated earlier, that one could, with the rationalized Runge–Kutta scheme, use very large time steps without a catastrophic instability. To be sure, the solutions underwent large oscillations when the time step was so large that the Courant condition was violated by a factor of 1000. Obviously a transient of this sort is without physical meaning. To obtain a true transient it is best not to violate the Courant condition by much, if at all.

Our calculations, carried out on a uniform 17×65 grid, took about 0.36 ms of CPU-time per grid point. The computer we used was an Amdahl 470 V/8. For the higher values of outside heat transfer coefficients a time step of 0.005 was deemed sufficiently small to obtain a true transient. We reduced this to 0.0001 for the lower values of the outside heat transfer coefficients. Computations on a 17×65 grid are sufficiently accurate to calculate Nusselt numbers to a few percent.

3. RESULTS AND DISCUSSION

3.1. *Flow structure*

For all cases to be reported we took the fluid in the cavity to be air with $Pr = 0.71$. By taking first $Nu_l = Nu_r = 100$ we obtained stream-patterns and isotherms qualitatively similar to those reported in ref. [1]. The flow has odd symmetry about the center point of the cavity.

The odd symmetry can be broken by letting the outside heat transfer coefficients have differing values. Thus by assigning the values $Nu_l = 300$ and $Nu_r = 1200$, we obtained the streamline and isotherm plots shown in Fig. 1. That the flow is no longer of odd symmetry is best seen from the multicellular stream-pattern at $Gr = 20\,000$. From the plot of the temperature of the right wall, in Fig. 2, it is clear that the transition to multicellular flow has taken place at $Gr = 15\,000$ or so. Since the right wall is the colder one of the two, its temperature is higher at the top, in a region which is heated by the hot fluid flowing across the top of the cavity. At $Gr = 5000$ in the main part of the flow the wall temperature is uniform, indicating that the flow is dominated by conduction. At $Gr = 40\,000$ the multicellular flow is shown to produce undulations in the temperature profile, the peaks of which are

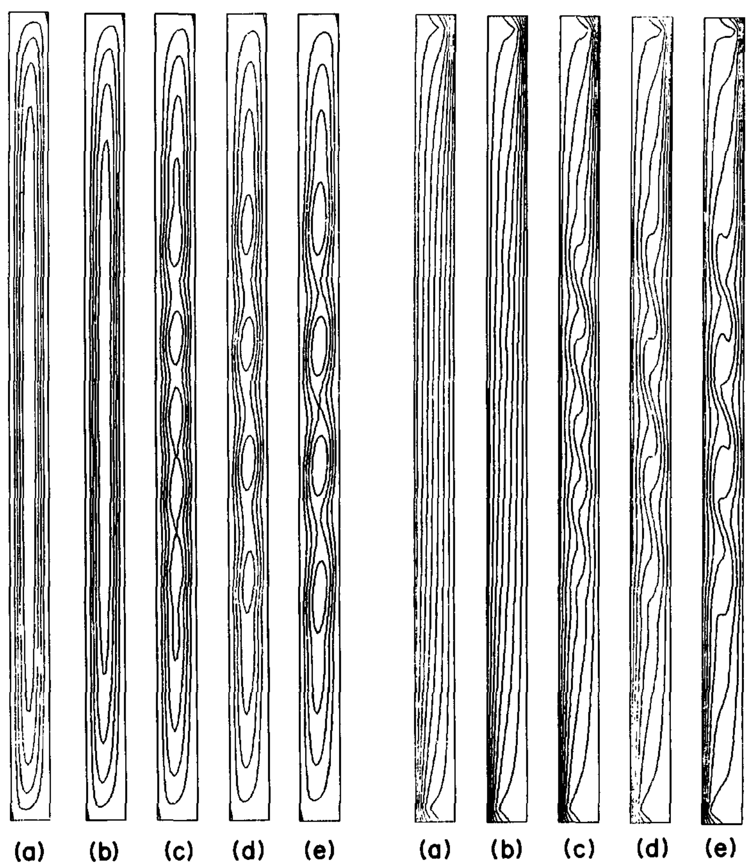

FIG. 1. Streamlines and isotherms; $A = 20$, $Pr = 0.71$, $Nu_l = 300$ and $Nu_r = 1200$. (a) $Gr = 10\,000$, (b) $Gr = 15\,000$, (c) $Gr = 20\,000$, (d) $Gr = 25\,000$, (e) $Gr = 40\,000$.

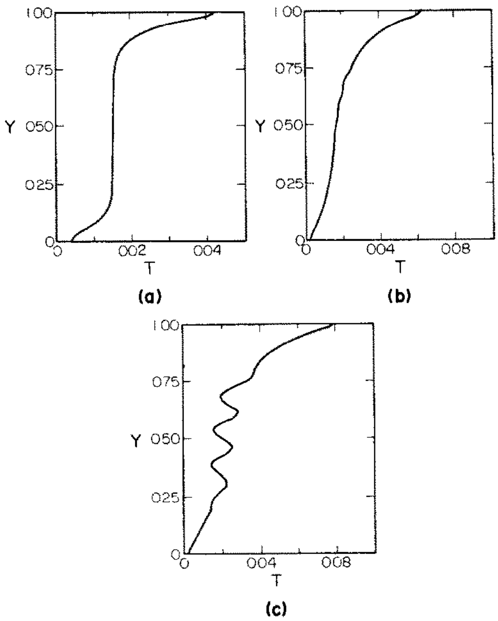

FIG. 2. Temperature distribution along the right wall; $A = 20$, $Pr = 0.71$, $Nu_l = 300$ and $Nu_r = 1200$. (a) $Gr = 5000$, (b) $Gr = 15\,000$, (c) $Gr = 40\,000$.

at locations where the hot fluid circulating in a cell comes to closest contact with the cold wall.

From earlier studies, and particularly from one by Bergholz [6], we know that the flow is stabilized by a stable vertical temperature gradient. Since outside convective boundary conditions allow the wall temperatures to adjust to this stratification, one expects that the flow in the case at hand is more stable than the one in a cavity with isothermal walls. By calculating the average wall temperatures and basing the Grashof number on the temperature difference between the walls one readily confirms this conjecture.

3.2. *Heat transfer*

The effect of the convective boundary conditions on the effective conductivity is shown in Table 1. The values of outside Nusselt numbers are representative of a small window interacting with the ambient through moderate heat transfer coefficients. In addition to the effective conductivity ratios the averaged nondimensional temperatures of the panes are shown. Thus the fractional temperature drop across the convective resistances is readily calculated. The temperature drop across the panes is neglected. The column of numbers for isothermal walls corresponds to the effective conductivity ratio of an air gap if the outside Nusselt numbers are taken to be infinitely large. For finite outside Nusselt numbers the effective conductivity drops, but it drops as a result of a lower average temperature difference across the panes and hence a weaker convective motion in the cavity. To compare the results on an equal basis, we calculated a Grashof number based on the difference in the average wall temperatures and obtained the effective conductivity ratio from a correlation based on the numbers in Table 3 of ref. [2]. These are listed in Table 2. They show that the effective conductivity of the air gap increases somewhat as a result of the difference in the convective pattern which develops because the wall temperatures now vary with height.

The error one commits by using correlations based on isothermal walls is listed in the third column. For the larger outside convective resistances the error is about 10%.

The structure of the flow changes if a constant heat flux is imposed, say, at the left wall while the right wall is held at a constant temperature. A plot of the streamlines and isotherms is shown in Fig. 3. This is a time sequence with $\Delta t = 0.1$

Table 2. Comparison of effective conductivity ratios for different values of Gr_w, $A = 20$ and $Pr = 0.71$

Gr_w	Isothermal walls	$Nu_r = 300$, $Nu_r = 1200$	Percentage deviation from isothermal walls (%)
9070	1.21	1.24	3
9955	1.23	1.27	3
10 848	1.28	1.30	2
13 470	1.37	1.38	1
17 780	1.48	1.53	3
22 100	1.58	1.61	2

Gr_w	Isothermal walls	$Nu_l = 100$, $Nu_r = 100$	Percentage deviation from isothermal walls (%)
8016	1.19	1.26	6
9780	1.22	1.34	10
12 680	1.34	1.46	9
15 450	1.42	1.56	10
23 440	1.60	1.79	12

Table 1. Effective conductivity ratios (k_{eff}/k) for isothermal walls and walls with convective boundary conditions for various Gr, $A = 20$ and $Pr = 0.71$

Grashof numbers	Isothermal walls	Convective boundary conditions					
		$Nu_l = 300$, $Nu_r = 1200$			$Nu_l = 100$, $Nu_r = 100$		
Gr	$Nu_w = k_{eff}/k$	Nu_w	$\bar{T}_{wl}$	$\bar{T}_{wr}$	Nu_w	$\bar{T}_{wl}$	$\bar{T}_{wr}$
5000	1.11	1.10	0.933	0.017	1.09	0.849	0.152
10 000	1.25	1.24	0.926	0.019	1.17	0.837	0.163
11 000	1.29	1.27	0.924	0.019	1.24	0.835	0.165
12 000	1.32	1.30	0.923	0.019	1.26	0.834	0.166
15 000	1.41	1.38	0.918	0.020	1.34	0.826	0.174
20 000	1.53	1.53	0.911	0.022	1.46	0.817	0.183
25 000	1.62	1.61	0.907	0.023	1.56	0.809	0.191
40 000	—	1.85	0.895	0.026	1.79	0.793	0.207
50 000	—	—	—	—	1.91	0.786	0.214

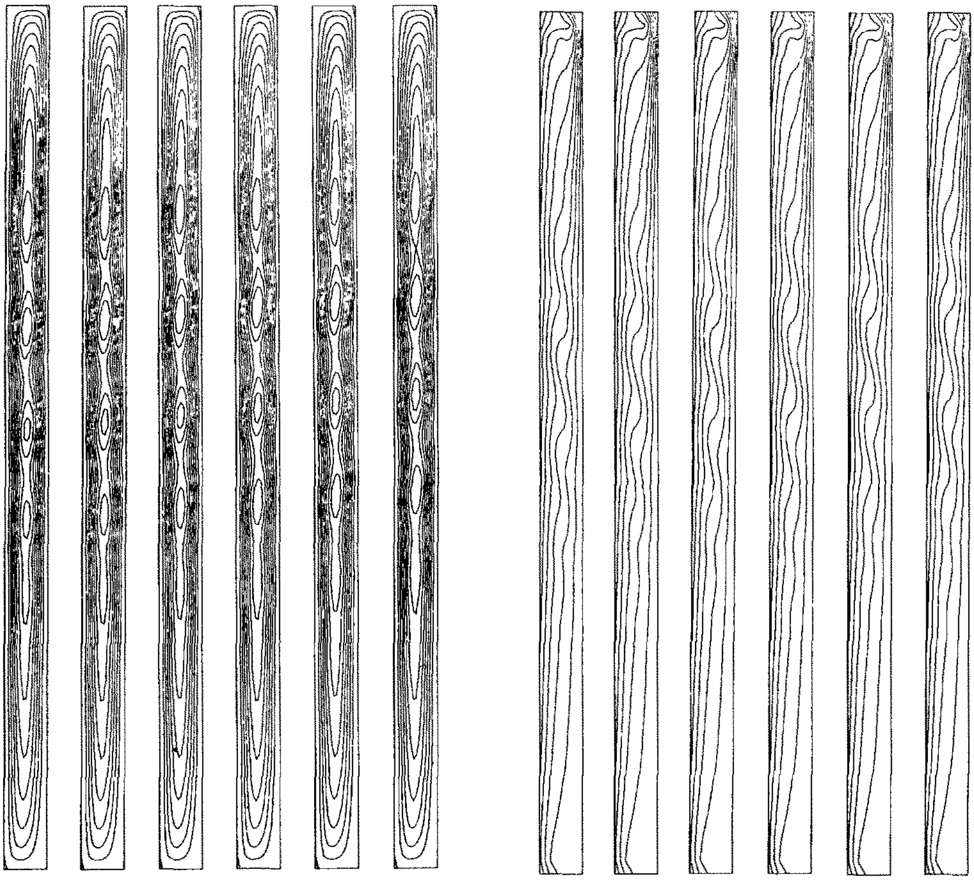

FIG. 3. The stream-pattern and isotherms of upward drifting cells for a flow in a cavity with an isothermal right wall and a constant heat flux on the left wall. $A = 20$, $Pr = 0.71$, $Gr = 24\,000$, $\Delta t = 0.1$.

between the frames. It shows that the cells are not stationary, but drift toward the top.

4. DISCUSSION

The convective boundary conditions are seen neither to change the flow structure nor the heat transfer by much. The wall temperature variations nevertheless bring about changes in the thermal structure which are manifest in the flow being now more stable. This is in contrast to the situation in which the heat flux is imposed on one side, for in that case the multicellular structure was seen to drift upward. We have discussed in refs. [3, 7] that breaking the odd symmetry of the problem, by making the sidewalls curved, as they are in an annulus, causes this upward drift. The symmetry can also be broken by imposing unsymmetrical thermal boundary conditions for a flow in a Cartesian geometry. In an infinitely tall annulus the drift speed depends continuously on the curvature. It may be, however, that for a cavity of finite aspect ratio, one needs to exceed a certain threshold value before such qualitative differences occur. Of course this need not be so. It is entirely possible that for the outside heat transfer coefficients considered here, the drift speed is so small that it is difficult to observe.

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